

LOGARITHMIC SOBOLEV INEQUALITY REVISITED

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ABSTRACT. We provide a new characterization of the logarithmic Sobolev inequality.

1. INTRODUCTION

The classical Sobolev inequality translates information about the derivatives of a function into information about the size of the function itself. Precisely, for a function u with square summable gradient in dimension N one obtains that u is $2N/(N-2)$ -summable, that is a gain in summability which depends on N and which tends to deteriorate as $N \rightarrow \infty$. On the other hand, since the middle fifties, people have started looking at possible replacements of the Sobolev inequality in order to provide an improvement in the summability *independent* of the dimension N , which can be done in terms of integrability properties of $u^2 \log u^2$. This was firstly done by Stam [23] who proved the logarithmic Sobolev inequality with Gauss measure $d\mathcal{G}$

$$\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_{2,d\mathcal{G}}^2} d\mathcal{G} \leq \frac{1}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 d\mathcal{G}, \quad d\mathcal{G} = e^{-\pi|x|^2} dx.$$

The formula was originally discovered in quantum field theory in order to handle estimates which are uniform in the space dimension, for systems with a large number of variables. A different proof and further insight was obtained by Gross in [17]. See also the work of Adams and Clarke [1] for an elementary proof of the previous inequality. These properties are widely used in statistical mechanics, quantum field theory and differential geometry. A variant of the logarithmic Sobolev inequality with Gauss measure is given by the following one parameter family of euclidean inequalities [18, Theorem 8.14]

$$\int_{\mathbb{R}^N} u^2 \log \frac{u^2}{\|u\|_2^2} dx + N(1 + \log a) \|u\|_2^2 \leq \frac{a^2}{\pi} \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

for any $u \in H^1(\mathbb{R}^N)$ and $a > 0$. A version of this inequality for fractional Sobolev spaces $H^s(\mathbb{R}^N)$ can be found in [13]. Recently some new characterization of the Sobolev spaces were provided in [4, 19, 21] (see also [2, 3, 5–9, 20]) in terms of the following family of nonlocal functionals

$$I_\delta(u) := \int \int_{\{|u(y)-u(x)|>\delta\}} \frac{\delta^2}{|x-y|^{N+2}} dx dy, \quad \delta > 0,$$

where u is a measurable function on \mathbb{R}^N . In particular, if $N \geq 3$ and $I_\delta(u) < \infty$ for some $\delta > 0$, then in [21] it was proved that

$$(1.1) \quad \int_{\{|u|>\lambda_N \delta\}} |u|^{2N/(N-2)} dx \leq C_N I_\delta(u)^{N/(N-2)},$$

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for some positive constants C_N and λ_N . This is a sort of nonlocal improvement of the classical Sobolev inequality and it is also possible to show that in the singular limit $\delta \searrow 0$ one recovers the classical Sobolev result, since I_δ converges to the Dirichlet energy up to a normalization constant. The aim of this note is to remark that in this context also a logarithmic type estimate holds. Thus we have the summability gain independent of N can be controlled in terms of $I_\delta(u)$.

More precisely, we have the following

Theorem 1.1. *Let $u \in L^2(\mathbb{R}^N)$ ($N \geq 3$). There is a positive constant C_N such that*

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^2 \leq \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} \|u\|_2^{\frac{2N-4}{N}} + C_N I_\delta(u) \right),$$

for all $\delta > 0$. In particular, if $u \in L^2(\mathbb{R}^N)$ is such that $I_\delta(u) < \infty$ for some $\delta > 0$, then

$$(1.2) \quad \int_{\mathbb{R}^N} u^2 \log u^2 dx < +\infty.$$

Proof. By a simple normalization argument, we may reduce the assertion to proving that

$$(1.3) \quad \int_{\mathbb{R}^N} u^2 \log u^2 dx \leq \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} + C_N I_\delta(u) \right), \quad \text{for all } \delta > 0,$$

for any $u \in L^2(\mathbb{R}^N)$ such that $\|u\|_2 = 1$. Considering the normalized outer measure

$$\mu(E) := \int_E u^2(x) dx, \quad \mu(\mathbb{R}^N) = 1,$$

and using Jensen's inequality for concave nonlinearities and with measure μ , we have

$$(1.4) \quad \begin{aligned} \log \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} dx \right) &= \log \left(\int_{\mathbb{R}^N} |u|^{\frac{4}{N-2}} d\mu \right) \geq \int_{\mathbb{R}^N} \log |u|^{\frac{4}{N-2}} d\mu \\ &= \frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 dx. \end{aligned}$$

On the other hand, applying (1.1), we derive that, for all $\delta > 0$,

$$\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 dx \leq \log \left(D_N \delta^{\frac{4}{N-2}} + C_N I_\delta(u)^{\frac{N}{N-2}} \right),$$

for some positive constant D_N , which implies (1.3). Here we used the fact that

$$\int_{\{|u| \leq \lambda_N \delta\}} |u|^{\frac{2N}{N-2}} dx \leq \lambda_N^{\frac{4}{N-2}} \delta^{\frac{4}{N-2}},$$

since $\int_{\mathbb{R}^N} u^2 dx = 1$. □

Defining a notion of *entropy* as typical in statistical mechanics:

$$\text{Ent}_\mu(f) := \int_{\mathbb{R}^N} \frac{f}{\|f\|_{1,\mu}} \log \frac{f}{\|f\|_{1,\mu}} d\mu + \frac{N}{2} \log \|f\|_{1,\mu}, \quad f \geq 0, \quad \|f\|_{1,\mu} := \int f d\mu,$$

the conclusion of the previous results reads as

$$u \in L^2(\mathbb{R}^N), \exists \delta > 0 : I_\delta(u) < +\infty \implies \text{Ent}_{\mathcal{L}^N}(u^2) < +\infty.$$

Remark 1.2 (Logarithmic NLS). If $u \in H^1(\mathbb{R}^N)$, then the results of [19] show that

$$(1.5) \quad \lim_{\delta \searrow 0} I_\delta(u) = Q_N \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

for some constant $Q_N > 0$. Hence, passing to the limit as $\delta \searrow 0$ in the inequality of Theorem 1.1 one recovers classical forms of the logarithmic inequality. The logarithmic Schrödinger equation

$$(1.6) \quad i\partial_t \phi + \Delta \phi + \phi \log |\phi|^2 = 0, \quad \phi : [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{C}, \quad N \geq 3,$$

admits applications to quantum mechanics, quantum optics, transport and diffusion phenomena, theory of superfluidity and Bose-Einstein condensation (see [25] and [10–12]). The *standing waves* solutions of (1.6) solve the following semi-linear elliptic problem

$$(1.7) \quad -\Delta u + \omega u = u \log u^2, \quad u \in H^1(\mathbb{R}^N).$$

These equations were recently investigated in [14, 24]. From a variational point of view, the search of solutions to (1.7) can be associated with the study of critical points (in a nonsmooth sense) of the lower semi-continuous functional $J : H^1(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx,$$

which is well defined by the logarithmic Sobolev inequality. Due to Theorem 1.1 and (1.5), one could handle a kind of *nonlocal approximations* of (1.7), formally defined for $\delta > 0$ by

$$I'_\delta(u) + \omega u = u \log u^2,$$

which are associated with the energy functional $J_\delta : H^1(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$J_\delta(u) = I_\delta(u) + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 dx.$$

Since there holds $I_\delta(u) \leq C_N \int_{\mathbb{R}^N} |\nabla u|^2 dx$ for all $\delta > 0$ and $u \in H^1(\mathbb{R}^N)$ (cf. [19, Theorem 2]) the energy functional J_δ is well defined, for every $\delta > 0$.

Remark 1.3 (Magnetic case). If $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is locally bounded and $u : \mathbb{R}^N \rightarrow \mathbb{C}$, we set

$$\Psi_u(x, y) := e^{i(x-y) \cdot A\left(\frac{x+y}{2}\right)} u(y), \quad x, y \in \mathbb{R}^N.$$

It was observed in [15] that the following *Diamagnetic inequality* holds

$$||u(x)| - |u(y)|| \leq |\Psi_u(x, x) - \Psi_u(x, y)|, \quad \text{for a.e. } x, y \in \mathbb{R}^N.$$

In turn, by defining

$$I_\delta^A(u) := \int_{\{|\Psi_u(x, y) - \Psi_u(x, x)| > \delta\}} \frac{\delta^2}{|x - y|^{N+2}} dx dy,$$

we have

$$(1.8) \quad I_\delta(|u|) \leq I_\delta^A(u), \quad \text{for all } \delta > 0 \text{ and all measurable } u : \mathbb{R}^N \rightarrow \mathbb{C}.$$

Then, Theorem 1.1 yields the following *Magnetic logarithmic Sobolev inequality*. For $u \in L^2(\mathbb{R}^N)$, there is a positive constant C_N such that

$$\int_{\mathbb{R}^N} \frac{|u|^2}{\|u\|_2^2} \log \frac{|u|^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^2 \leq \frac{N}{2} \log \left(C_N \delta^{\frac{4}{N}} \|u\|_2^{\frac{2N-4}{N}} + C_N I_\delta^A(u) \right).$$

Notice that, since $I_\delta(|u|) \approx \|\nabla |u|\|_2^2$ as $\delta \searrow 0$ [19] and $I_\delta^A(u) \approx \|\nabla u - iAu\|_2^2$ as $\delta \searrow 0$ [22], from inequality (1.8) one recovers $\|\nabla |u|\|_2 \leq \|\nabla u - iAu\|_2$ which follows from the well-know diamagnetic inequality for the gradients $|\nabla |u|| \leq |\nabla u - iAu|$, see [18].

As a companion to Theorem 1.1, we also have the following

Theorem 1.4. *Let $u \in L^2(\mathbb{R}^N)$ ($N \geq 3$). Assume that there exists a non-decreasing function $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $F(ts) \leq t^\beta F(s)$ for any $s, t \geq 0$ and some $\beta > 0$ and*

$$(1.9) \quad \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy < +\infty.$$

Then there exists a positive constant $C_{N,F}$ such that

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} dx + \frac{N}{2} \log \|u\|_2^\beta \leq \frac{N}{2} \log \left(C_{N,F} \|u\|_2^\beta + C_{N,F} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right),$$

In particular, condition (1.2) holds.

Proof. Consider the statement when $\|u\|_2 = 1$. In light of inequality (1.4), since by [21, Proposition 6] there exists $C_N > 0$ and $\lambda_N > 0$ such that

$$(1.10) \quad \int_{\{|u| > \lambda_N F(1/2)\}} |u|^{2N/(N-2)} dx \leq C_N \left(\frac{1}{F(1/2)} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right)^{N/(N-2)},$$

by arguing as in the previous proof, we get

$$\frac{2}{N-2} \int_{\mathbb{R}^N} u^2 \log u^2 \leq \log \left(D_{N,F} + D_{N,F} \left(\int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right)^{N/(N-2)} \right),$$

where we used the fact that

$$\int_{\{|u| \leq \lambda_N F(1/2)\}} |u|^{\frac{2N}{N-2}} dx \leq \lambda_N^{\frac{4}{N-2}} F(1/2)^{\frac{4}{N-2}},$$

since $\int_{\mathbb{R}^N} u^2 dx = 1$. Then, we get

$$\int_{\mathbb{R}^N} u^2 \log u^2 \leq \frac{N}{2} \log \left(C_{N,F} + C_{N,F} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right).$$

In the general case, using the sub-homogeneity condition on F yields

$$\int_{\mathbb{R}^N} \frac{u^2}{\|u\|_2^2} \log \frac{u^2}{\|u\|_2^2} \leq \frac{N}{2} \log \left(C_{N,F} + \frac{C_{N,F}}{\|u\|_2^\beta} \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+2}} dx dy \right),$$

which yields the desired conclusion. \square

Remark 1.5 ($L^p(\mathbb{R}^N)$ -version). If $p > 1$ and $N > p$, one has a variant of (1.4), namely

$$(1.11) \quad \log \left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}} dx \right) \geq \frac{p}{N-p} \int_{\mathbb{R}^N} |u|^p \log |u|^p dx.$$

Then, by arguing as in the proofs of Theorems 1.1 and 1.4 with

$$(1.12) \quad u \mapsto \int \int_{\{|u(y) - u(x)| > \delta\}} \frac{\delta^p}{|x - y|^{N+p}} dx dy, \quad u \mapsto \int_{\mathbb{R}^{2N}} \frac{F(|u(x) - u(y)|)}{|x - y|^{N+p}} dx dy,$$

in place of $I_\delta(u)$ and (1.9) respectively, it is possible to get corresponding log-Sobolev inequalities as for the case $p = 2$, via the results of [21]. In particular, if $u \in L^p(\mathbb{R}^N)$ and the functionals in (1.12) are finite at u for some $\delta > 0$, then

$$\int_{\mathbb{R}^N} |u|^p \log |u|^p dx < +\infty.$$

The Euclidean logarithmic Sobolev inequalities for the p -case have been intensively studied, see e.g. the work of Del Pino and Dolbeault [16] and the references therein.

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